

## SPM BULLETIN

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## 1. EDITOR'S NOTE

After a long break, we are back with some very interesting research announcements and an open problem which is one of the most difficult, long lasting, and important problems in the field.

A major change in this bulletin is that from now on it will usually not appear monthly, but more close to quarterly. Special announcements (if urgent) will be made by text emails. Contributions to the next issue are, as always, welcome.

**Previous issues.** The first issues of this bulletin, which contain general information (first issue), basic definitions, research announcements, and open problems (all issues) are available online:

- (1) First issue: <http://arxiv.org/abs/math.GN/0301011>
- (2) Second issue: <http://arxiv.org/abs/math.GN/0302062>
- (3) Third issue: <http://arxiv.org/abs/math.GN/0303057>
- (4) Fourth issue: <http://arxiv.org/abs/math.GN/0304087>

(5) Fifth issue: <http://arxiv.org/abs/math.GN/0305367>

**Contributions.** Please submit your contributions (announcements, discussions, and open problems) by e-mailing us. It is preferred to write them in  $\text{\LaTeX}$ . The authors are urged to use as standard notation as possible, or otherwise give a reference to where the notation is explained. Contributions to this bulletin would not require any transfer of copyright, and material presented here can be published elsewhere.

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## 2. RESEARCH ANNOUNCEMENTS

**2.1. The Topological Version of Fodor's Theorem.** The following purely topological generalization is given of Fodor's theorem (also known as the "pressing down lemma"): Let  $X$  be a locally compact, non-compact  $T_2$  space such that any two closed unbounded<sup>1</sup> (cub) subsets of  $X$  intersect; call  $S \subset X$  stationary if it meets every cub in  $X$ . Then for every neighbourhood assignment  $U$  defined on a stationary set  $S$  there is a stationary subset  $T \subset S$  such that

$$\cap \{U(x) : x \in T\} \neq \emptyset.$$

Just like the "modern" proof of Fodor's theorem, our proof hinges on a notion of diagonal intersection of cub's, definable under some additional conditions.

We also use these results to present an (alas, only partial) generalization to this framework of Solovay's celebrated stationary set decomposition theorem.

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**2.2. Continuous images of big sets and additivity of  $s_0$  under  $\text{CPA}_{\text{prism}}$ .** We prove that the Covering Property Axiom  $\text{CPA}_{\text{prism}}$ , which holds in the iterated perfect set model, implies the following facts.

- There exists a family  $\mathcal{G}$  of uniformly continuous functions from  $\mathbb{R}$  to  $[0, 1]$  such that  $|\mathcal{G}| = \omega_1$  and for every  $S \in [\mathbb{R}]^{\mathfrak{c}}$  there exists a  $g \in \mathcal{G}$  with  $g[S] = [0, 1]$ .
- The additivity of the Marczewski ideal  $s_0$  of is equal to  $\omega_1 < \mathfrak{c}$ .

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<sup>1</sup>A set is bounded if it has compact closure.

**2.3. Uncountable  $\gamma$ -sets under axiom  $\text{CPA}_{\text{cube}}^{\text{game}}$ .** In the paper we formulate a Covering Property Axiom  $\text{CPA}_{\text{cube}}^{\text{game}}$ , which holds in the iterated perfect set model, and show that it implies the existence of uncountable strong  $\gamma$ -sets in  $\mathbb{R}$  (which are strongly meager) as well as uncountable  $\gamma$ -sets in  $\mathbb{R}$  which are not strongly meager. These sets must be of cardinality  $\omega_1 < \mathfrak{c}$ , since every  $\gamma$ -set is universally null, while  $\text{CPA}_{\text{cube}}^{\text{game}}$  implies that every universally null has cardinality less than  $\mathfrak{c} = \omega_2$ .

We will also show that  $\text{CPA}_{\text{cube}}^{\text{game}}$  implies the existence of a partition of  $\mathbb{R}$  into  $\omega_1$  null compact sets.

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**2.4. Ultrafilters with property (s).** A set  $X$  which is a subset of the Cantor set has property (s) (Marczewski (Spzilrajn)) iff for every perfect set  $P$  there exists a perfect set  $Q$  contained in  $P$  such that  $Q$  is a subset of  $X$  or  $Q$  is disjoint from  $X$ . Suppose  $U$  is a nonprincipal ultrafilter on  $\omega$ . It is not difficult to see that if  $U$  is preserved by Sacks forcing, i.e., it generates an ultrafilter in the generic extension after forcing with the partial order of perfect sets, then  $U$  has property (s) in the ground model. It is known that selective ultrafilters or even  $P$ -points are preserved by Sacks forcing. On the other hand (answering a question raised by Hrusak) we show that assuming CH (or more generally MA for countable posets) there exists an ultrafilter  $U$  with property (s) such that  $U$  does not generate an ultrafilter in any extension which adds a new subset of  $\omega$ .

The paper is available online at

<http://arXiv.org/abs/math/0310438>

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**2.5. There may be no Hausdorff ultrafilters.** An ultrafilter  $U$  is Hausdorff if for any two functions  $f, g$  mapping  $\mathbb{N}$  to  $\mathbb{N}$ ,  $f(U) = g(U)$  iff  $f(n) = g(n)$  for  $n$  in some  $X$  in  $U$ . We will show that it is consistent that there are no Hausdorff ultrafilters.

The paper is available online at

<http://arXiv.org/abs/math/0311064>

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**2.6. Proper forcing and rectangular Ramsey theorems.** I prove forcing preservation theorems for products of definable partial orders preserving the cofinality of the meager or null ideal. Rectangular Ramsey theorems for related ideals follow from the proofs.

The paper is available online at

<http://arXiv.org/abs/math/0311135>

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**2.7. Cardinal characteristics of the ideal of Haar null sets.** We calculate the cardinal characteristics of the  $\sigma$ -ideal  $\mathcal{HN}(G)$  of Haar null subsets of a Polish non-locally compact group  $G$  with invariant metric and show that  $\text{cov}(\mathcal{HN}(G)) \leq \mathfrak{b} \leq \max\{\mathfrak{d}, \text{non}(\mathcal{N})\} \leq \text{non}(\mathcal{HN}(G)) \leq \text{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \text{non}(\mathcal{N})\}$ . If  $G = \prod_{n \geq 0} G_n$  is the product of abelian locally compact groups  $G_n$ , then  $\text{add}(\mathcal{HN}(G)) = \text{add}(\mathcal{N})$ ,  $\text{cov}(\mathcal{HN}(G)) = \min\{\mathfrak{b}, \text{cov}(\mathcal{N})\}$ ,  $\text{non}(\mathcal{HN}(G)) = \max\{\mathfrak{d}, \text{non}(\mathcal{N})\}$  and  $\text{cof}(\mathcal{HN}(G)) \geq \text{cof}(\mathcal{N})$ , where  $\mathcal{N}$  is the ideal of Lebesgue null subsets on the real line. Martin Axiom implies that  $\text{cof}(\mathcal{HN}(G)) > 2^{\aleph_0}$  and hence  $G$  contains a Haar null subset of  $G$  that cannot be enlarged to a Borel or projective Haar null subset of  $G$ . This gives a negative (consistent) answer to a question of S.Solecki. To obtain these estimates we show that for a Polish non-locally compact group  $G$  with invariant metric the ideal  $\mathcal{HN}(G)$  contains all  $\sigma$ -bounded subsets (equivalently, subsets with the small ball property) of  $G$ .

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**2.8. A note on generalized Egorov's theorem.** We prove that the following generalized version of Egorov's theorem is independent from the ZFC axioms of the set theory.

Let  $\{f_n\}_{n \in \omega}$ ,  $f_n : \langle 0, 1 \rangle \rightarrow R$ , be a sequence of functions (not necessarily measurable) converging pointwise to zero for every  $x \in \langle 0, 1 \rangle$ . Then for every  $\varepsilon > 0$ , there are a set  $A \subseteq \langle 0, 1 \rangle$  of Lebesgue outer measure  $m^* > 1 - \varepsilon$  and a sequence of integers  $\{n_k\}_{k \in \omega}$  with  $\{f_{n_k}\}_{k \in \omega}$  converging uniformly on  $A$ .

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### 3. PROBLEM OF THE MONTH

The following is extracted from [5]. Borel's Conjecture, which was proved to be consistent by Laver, implies that each set of reals satisfying  $\mathcal{S}_1(\mathcal{O}, \mathcal{O})$  (and the classes below it) is countable. From our point of view this means that there do not exist ZFC examples of sets satisfying  $\mathcal{S}_1(\mathcal{O}, \mathcal{O})$ . A set of reals  $X$  is a  $\sigma$ -set if each  $G_\delta$  set in  $X$  is also an  $F_\sigma$  set in  $X$ . In [4] it is proved that every element of  $\mathcal{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$  is a  $\sigma$ -set. According to a result of Miller [2], it is consistent that every  $\sigma$ -set of real numbers is countable. Thus, there do not exist uncountable ZFC examples satisfying  $\mathcal{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ . The situation for the other classes, though addressed by top experts, remains open. In particular, we have the following.

**Problem 3.1** ([3], [4, Problem 45], [1]). *Does there exist (in ZFC) an uncountable set of reals satisfying  $\mathcal{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$ ?*

By [4], this is the same as asking whether it is consistent that each uncountable set of reals can be mapped onto a dominating subset of  ${}^{\mathbb{N}}\mathbb{N}$  by a Borel function. This is one of the major open problems in the field.

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